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# On the equivariant reduction of structure group of a principal bundle to a Levi subgroup

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**Abstract**

Let $M$ be an irreducible projective variety, defined over an algebraically closed field $k$ of characteristic zero, equipped with an action of a group $\Gamma$. Let $E_G$ be a principal $G$-bundle over $M$, where $G$ is a connected reductive linear algebraic group defined over $k$, equipped with a lift of the action of $\Gamma$ on $M$. We give conditions for $E_G$ to admit a $\Gamma$-equivariant reduction of structure group to $H$, where $H \subset G$ is a Levi subgroup. We show that for any principal $G$-bundle $E_G$, there is a naturally associated conjugacy class of Levi subgroups of $G$. Given a Levi subgroup $H$ in this conjugacy class, the principal $G$-bundle $E_G$ admits a $\Gamma$-equivariant reduction of structure group to $H$, and furthermore, such a reduction is unique up to an automorphism of $E_G$ that commutes with the action of $\Gamma$ on $E_G$.


**Résumé**

Soit $M$ une variété projective irréductible, définie sur un corps $k$ algébriquement clos de caractéristique nulle, munie d'une action de groupe $\Gamma$. Soit $E_G$ un $G$-fibré principal sur $M$, où $G$ est un groupe algébrique linéaire réductif connexe défini sur $k$, muni d'une action de $\Gamma$ relevant l'action sur $M$. Nous donnons des conditions pour que $E_G$ admette une réduction $\Gamma$-équivariante du groupe structural à $H$, où $H \subset G$ est un sous-groupe de Lévi. Nous montrons qu'à tout $G$-fibré principal $E_G$ est naturellement associée une classe de conjugaison de sous-groupes de Lévi de $G$. Étant donné un sous-groupe de Lévi $H$ dans cette classe, le fibré principal $E_G$ admet une réduction $\Gamma$-équivariante de son groupe structural à $H$ et, de plus, une telle réduction est unique à un automorphisme près de $E_G$, commutant à l'action de $\Gamma$.


*MSC:* 55R91; 14L30

*Keywords:* Principal bundle; Levi subgroup; Automorphism group; Reduction

## 1. Introduction

A holomorphic principal $G$-bundle over the complex projective line $\mathbb{CP}^1$ admits a holomorphic reduction of structure group to a maximal torus of $G$, where $G$ is a complex reductive group [5]. In particular, any holomorphic vector

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bundle over $\mathbb{CP}^1$ splits holomorphically as a direct sum of line bundles. If $T_{\mathbb{C}}$ is a complex torus acting algebraically on $\mathbb{CP}^1$ and $V$ a holomorphic vector bundle over $\mathbb{CP}^1$ equipped with a lift, as vector bundle automorphisms, of the action of $T_{\mathbb{C}}$ on $\mathbb{CP}^1$, then $V$ decomposes as a direct sum of holomorphic line subbundles with each line subbundle left invariant by the action of the torus [7, p. 500, Theorem]. In [4], an alternative proof of this theorem was given.

We will briefly recall the main result of [4], the predecessor of this paper. Let $k$ be an algebraically closed field of characteristic zero. Let $M$ be an irreducible projective variety defined over $k$ and $E_G$ a principal $G$-bundle over $M$, where $G$ is a connected reductive linear algebraic group defined over $k$. Assume that both $E_G$ and $M$ are equipped with algebraic actions of a connected algebraic group $S$ defined over $k$. The action of $S$ on $E_G$ commutes with the action of $G$, and the projection of $E_G$ to $M$ is $S$-equivariant. Assume that the $G$-bundle $E_G$ admits a reduction of structure group to a maximal torus $T$ of $G$. Then $E_G$ admits a $S$-equivariant reduction of structure group to $T$ if and only if the action of $S$ on the automorphism group of $E_G$ leaves a maximal torus invariant [4, p. 127, Theorem 2.1].

A maximal torus of $G$ is a special case of a Levi subgroup of $G$. The aim here is to investigate conditions under which a principal $G$-bundle over a projective variety equipped with an algebraic action of an abstract group $\Gamma$ admits a $\Gamma$-equivariant reduction of structure group to a Levi subgroup of $G$.

Let $\Gamma$ be an abstract group. Let $M$ be an irreducible projective variety defined over an algebraically closed field $k$ of characteristic zero on which the group $\Gamma$ acts as algebraic automorphisms. Let $G$ be a connected reductive linear algebraic group defined over the field $k$ and $E_G$ a principal $G$-bundle over $M$. Let $\mathrm{Aut}(E_G)$ denote the group of all automorphisms of $E_G$. We will assume that the principal $G$-bundle $E_G$ is equipped with a lift of the action of $\Gamma$ on $M$ that commutes with the action of $G$ on $E_G$. More precisely, the automorphism of the total space of $E_G$ defined by any $\gamma \in \Gamma$ is an algebraic automorphism of the $G$-bundle over the automorphism of $M$ defined by $\gamma$. The action of $\Gamma$ on $E_G$ induces an action of $\Gamma$ on $\mathrm{Aut}(E_G)$. The induced action of $\Gamma$ on $\mathrm{Aut}(E_G)$ preserves the group structure of $\mathrm{Aut}(E_G)$.

A torus is a product of copies of the multiplicative group $\mathbb{G}_m$, or it is the trivial group. By a Levi subgroup of $G$ we will mean the centralizer of some torus of $G$. Let,

$$E_H \subset E_G, \tag{1.1}$$

be a reduction of structure group of the principal $G$-bundle $E_G$ to a Levi subgroup $H$ of $G$. We will denote by $Z_0(H)$ the connected component of the center of $H$ containing the identity element. So $Z_0(H)$ is contained in the automorphism group of $E_H$ and hence it is contained in $\mathrm{Aut}(E_G)$.

We prove the following criterion for $E_H$ to be left invariant by the action of $\Gamma$ on $E_G$.

**Theorem 1.1** (Theorem 2.2). *The principal $H$-bundle $E_H$ in* (1.1) *is left invariant by the action of $\Gamma$ on $E_G$ if and only if $\Gamma$ acts trivially on the subgroup $Z_0(H) \subset \mathrm{Aut}(E_G)$.*

In Theorem 1.1 we fix the reduction and ask if it is $\Gamma$-equivariant (that is, left invariant by the action of $\Gamma$ on $E_G$). On the other hand, we may fix only the Levi subgroup $H \subset G$ and ask whether there is any $\Gamma$-equivariant reduction of structure group of $E_G$ to $H$. In Lemma 3.2 and Proposition 3.3 we address this question and produce the following criterion.

**Proposition 1.2.** *A torus of $\mathrm{Aut}(E_G)$ determines a torus, unique up to inner automorphism, of $G$. The $G$-bundle $E_G$ admits a $\Gamma$-equivariant reduction of structure group to the Levi subgroup $H$ if and only if there is a torus $T \subset \mathrm{Aut}(E_G)$ satisfying the following two conditions*:

(1) *the action of $\Gamma$ on $T$ is trivial* (*recall that $\Gamma$ acts on* $\mathrm{Aut}(E_G)$);
(2) *there is a subtorus $T' \subset Z_0(H) \subset G$ in the conjugacy class of tori of $G$ defined by $T$ such that the centralizer of $T'$ in $G$ coincides with $H$.*

Let $T_0 \subset G$ be a torus in the conjugacy class of tori defined by a maximal torus of $\mathrm{Aut}(E_G)^\Gamma$, where $\mathrm{Aut}(E_G)^\Gamma \subset \mathrm{Aut}(E_G)$ is the subgroup fixed pointwise by the action of $\Gamma$ on $\mathrm{Aut}(E_G)$. The conjugacy class of $T_0$ does not depend on the choice of the maximal torus. Let $H_0 \subset G$ be the centralizer of $T_0$.

With the above notation, we prove the following theorem:

**Theorem 1.3** (Theorem 4.1). *The principal $G$-bundle $E_G$ admits a $\Gamma$-equivariant reduction of structure group $E_{H_0} \subset E_G$ to $H_0$, which is unique in the following sense*:

(1) *for any $\Gamma$-equivariant reduction of structure group $E'_{H_0} \subset E_G$ to $H_0$, there is an automorphism $\tau \in \mathrm{Aut}(E_G)^\Gamma$ such that $\tau(E_{H_0}) = E'_{H_0}$ as subvarieties of $E_G$*;
(2) *if $E_H$ is a $\Gamma$-equivariant reduction of structure group to a Levi subgroup $H \subset G$, then there is $g \in G$ and $\tau \in \mathrm{Aut}(E_G)^\Gamma$ such that $g^{-1}H_0 g \subset H$ and $E_{H_0}g \subset \tau(E_H)$.*

A theorem due to Atiyah says that for an isomorphism of a vector bundle over $M$ with any direct sum of indecomposable vector bundles, the direct summands are unique up to a permutation of the summands. In [3] it was proved that a principal $G$-bundle $E_G$ admits a reduction of structure to a Levi subgroup of $G$ determined by a maximal torus in $\mathrm{Aut}(E_G)$.

In the present paper we give a construction of reduction of structure group of a principal $G$-bundle equipped with a semisimple automorphism. We will briefly describe below this construction of reduction of structure group.

Let $E_G$ be a principal $G$-bundle over $M$. Let $\mathrm{Ad}(E_G) = (E_G \times G)/G$ be the adjoint bundle. We recall that the action of $G$ on $E_G \times G$, in the definition of $\mathrm{Ad}(E_G)$, is the following: any point $g \in G$ sends any point $(z,h) \in E_G \times G$ to $(zg, g^{-1}hg) \in E_G \times G$. Let,

$$F : E_G \times G \to \mathrm{Ad}(E_G), \tag{1.2}$$

be the quotient morphism. Since the conjugation action of $G$ on itself preserves the group structure of $G$, the fibers of $\mathrm{Ad}(E_G)$ are equipped with a group operation. From the definition of $\mathrm{Ad}(E_G)$ it follows immediately that for each point $x \in M$, the fiber $\mathrm{Ad}(E_G)_x$ is identified with $G$ up to an inner conjugation. In other words, there is a canonical bijective correspondence between the conjugacy classes in $G$ and the conjugacy classes in $\mathrm{Ad}(E_G)_x$.

Let $\phi : M \to \mathrm{Ad}(E_G)$ be a section of the fiber bundle $\mathrm{Ad}(E_G)$ over $M$ with the following property: for each point $x \in M$, the element $\phi(x) \in \mathrm{Ad}(E_G)_x$ is semisimple. In view of the above remark that conjugacy classes in $G$ are identified with the conjugacy classes in $\mathrm{Ad}(E_G)_x$, the element $\phi(x)$ defines a conjugacy class of semisimple elements in $G$. This conjugacy class in $G$ is actually independent of $x$. To see this note that if $E_V$ is a vector bundle over $M$, and

$$\sigma \in H^0\big(M, \mathrm{End}(E_V)\big)$$

an endomorphism of $E_V$, then the characteristic polynomial of $\sigma$ is a polynomial with coefficients in the ring of function on $M$. Since $M$ is a projective variety, any function on $M$ is a constant function. Hence the eigenvalues of $\sigma(x)$ is independent of $x$. Using this it is straightforward to deduce that the conjugacy class in $G$ defined by the semisimple element $\phi(x)$ is independent of $x$.

Fix an element $g_0 \in G$ in the conjugacy class in $G$ defined by $\phi(x)$. Set:

$$\widehat{E}_\phi := F^{-1}\big(\phi(M)\big) \cap \big(E_G \times \{g_0\}\big) \subset E_G \times G,$$

where $F$ is the projection in (1.2). Define:

$$E'_\phi := p_1\big(\widehat{E}_\phi\big) \subset E_G, \tag{1.3}$$

where $p_1$ is the natural projection of $E_G \times G$ to $E_G$. The subvariety $E'_\phi \subset E_G$ defined in (1.3) gives a reduction of structure group of the principal $G$-bundle $E_G$ to the centralizer of $g_0$ in $G$.

The main constructions done in this paper are based on this observation and the fact that any two maximal tori in an algebraic group defined over $k$ are conjugate. We note that when $G = \mathrm{GL}(n,k)$, then the above construction of reduction of structure group corresponds to the eigenbundle decomposition of a vector bundle $E$ over $M$ equipped with an automorphism of $E$.

A section of the fiber bundle $\mathrm{Ad}(E_G)$ over $M$ gives an automorphism of the principal $G$-bundle $E_G$. If $\phi$ is an arbitrary section of $\mathrm{Ad}(E_G)$ (not necessarily pointwise semisimple), then using the Jordan decomposition $\phi$ gives a section of $\mathrm{Ad}(E_G)$ which is pointwise semisimple. Therefore, the above construction of a reduction of structure group works for an arbitrary section of $\mathrm{Ad}(E_G)$. We hope that this construction of reduction of structure group will find use elsewhere also.

## 2. Equivariant reduction to a Levi subgroup

Let $M$ be an irreducible projective variety defined over an algebraically closed field $k$ of characteristic zero. Let $\Gamma$ be a group acting on the left of $M$. So we have a map,

$$\phi : \Gamma \times M \to M, \tag{2.1}$$

such that for any $\gamma \in \Gamma$, the map defined by $x \mapsto \phi(\gamma, x)$ is an algebraic automorphism of $M$, and furthermore, $\phi(\gamma_1\phi(\gamma_2, x)) = \phi(\gamma_1\gamma_2, x)$ for all $\gamma_1, \gamma_2 \in \Gamma$ and $x \in M$, with $\phi(e, x) = x$, where $e \in \Gamma$ is the identity element.

Let $G$ be a connected reductive linear algebraic group defined over the field $k$ and $E_G$ a principal $G$-bundle over $M$. Let,

$$f : E_G \to M, \tag{2.2}$$

be the projection. We will denote by $\mathrm{Aut}(E_G)$ the group of all automorphisms of the $G$-bundle $E_G$ (over the identity automorphism of $M$). So, $f(\tau(z)) = f(z)$ and $\tau(z)g = \tau(zg)$ for all $\tau \in \mathrm{Aut}(E_G)$ and all $z \in E_G$. Note that $\mathrm{Aut}(E_G)$ is an affine algebraic group defined over $k$. After fixing a faithful representation of $G$, the group $\mathrm{Aut}(E_G)$ gets identified with a closed subgroup of the automorphism group of the associated vector bundle; the automorphism group of a vector bundle is a Zariski open dense subset in the vector space defined by the space of all global endomorphisms of the vector bundle. The group $\mathrm{Aut}(E_G)$ is, in fact, the space of all global section of the adjoint bundle:

$$\mathrm{Ad}(E_G) := E_G \times^G G = (E_G \times G)/G,$$

(the action of any $g \in G$ sends any point $(z, g') \in E_G \times G$ to $(zg, g^{-1}g'g)$). Let,

$$\mathrm{Aut}^0(E_G) \subset \mathrm{Aut}(E_G),$$

be the connected component containing the identity element. So $\mathrm{Aut}^0(E_G)$ is a connected affine algebraic group defined over $k$.

Assume that $E_G$ is equipped with a lift of the action of $\Gamma$ on $M$. So the map,

$$\Phi : \Gamma \times E_G \to E_G, \tag{2.3}$$

defining the action has the property that for any $\gamma \in \Gamma$ the map defined by $z \mapsto \Phi(\gamma, z)$ is an algebraic automorphism of $E_G$ that commutes with the action of $G$ on $E_G$, and $f \circ \Phi(\gamma, z) = \phi(\gamma, f(z))$, where $f$ is as in (2.2). Note that the action of $\Gamma$ on $E_G$ induces an action of $\Gamma$ on $\mathrm{Aut}(E_G)$ through algebraic group automorphisms. More precisely, the action of any $\gamma \in \Gamma$ sends $F \in \mathrm{Aut}(E_G)$ to the automorphism defined by:

$$z \mapsto \Phi\big(\gamma, F\big(\Phi(\gamma^{-1}, z)\big)\big),$$

where $\Phi$ is as in (2.3).

The group $\Gamma$ acts on the adjoint bundle $\mathrm{Ad}(E_G)$ as follows: the action of any $\gamma \in \Gamma$ sends $(z, g) \in E_G \times G$ to:

$$\big(\Phi(\gamma, z), g\big) \in E_G \times G; \tag{2.4}$$

this descends to an action of $\Gamma$ on the quotient $\mathrm{Ad}(E_G) = (E_G \times G)/G$. This descended action of $\Gamma$ on $\mathrm{Ad}(E_G)$ lifts the action of $\Gamma$ on $M$, and it clearly preserves the algebraic group structure of the fibers of $\mathrm{Ad}(E_G)$. The action of $\Gamma$ on $\mathrm{Ad}(E_G)$ induces an action of $\Gamma$ on the space of all global sections of $\mathrm{Ad}(E_G)$, namely $\mathrm{Aut}(E_G)$. It is straightforward to check that this induced action on $\mathrm{Aut}(E_G)$ coincides with the earlier defined action of $\Gamma$ on $\mathrm{Aut}(E_G)$.

Let,

$$\mathrm{Aut}(E_G)^{\Gamma} \subset \mathrm{Aut}(E_G), \tag{2.5}$$

be the subgroup that is fixed pointwise by the action of $\Gamma$ on $\mathrm{Aut}(E_G)$. Since the action of each $\gamma \in \Gamma$ is an algebraic automorphism of $\mathrm{Aut}(E_G)$, the subgroup $\mathrm{Aut}(E_G)^{\Gamma}$ is Zariski closed.

A reduction of structure group of $E_G$ to a closed subgroup $H \subset G$ is a section of $E_G/H$ over $M$, or equivalently, a closed subvariety $E_H \subset E_G$ closed under the action of $H$ such that the $H$ action on $E_H$ defines a principal $H$-bundle over $M$.

**Definition 2.1.** A reduction of structure group $E_H \subset E_G$ to $H$ is called *$\Gamma$-equivariant* if the subvariety $E_H$ is left invariant by the action of $\Gamma$ on $E_G$.

Since the actions of $\Gamma$ and $G$ on $E_G$ commute, there is an induced action of $\Gamma$ on $E_G/H$. Evidently $E_H$ is a $\Gamma$-equivariant reduction of structure group if and only if the section over $M$ of the bundle $E_G/H \to M$ defined by $E_H$ is fixed by the action of $\Gamma$ on the space of all sections of $E_G/H$ induced by the action on $E_G/H$.

By a Levi subgroup of $G$ we mean the centralizer in $G$ of some torus of $G$. Recall that a torus is a product of copies of $\mathbb{G}_m$ or it is the trivial group. For a Levi subgroup $H \subset G$, the centralizer in $G$ of the connected component of the center of $H$ containing the identity element coincides with $H$ (see [9, §3]). If $H \subset G$ is a Levi subgroup, then there is a parabolic subgroup $H \subset P \subset G$ such that $H$ projects isomorphically to the Levi quotient of $P$. Conversely, if $H$ is a reductive subgroup of a parabolic subgroup $P \subset G$ such that $H$ projects isomorphically to the Levi quotient of $P$, then $H$ is a Levi subgroup of $G$. Note that if we take the torus to be the trivial group, then the corresponding Levi subgroup is $G$ itself, and hence in that case the corresponding parabolic subgroup is $G$.

Take a Levi subgroup $H \subset G$. Let,

$$Z_0(H) \subset H,$$

be the connected component of the center of $H$ containing the identity element. Let,

$$E_H \subset E_G, \tag{2.6}$$

be a reduction of structure group of $E_G$ to $H$. We have:

$$Z_0(H) \subset \mathrm{Aut}^0(E_H) \subset \mathrm{Aut}^0(E_G), \tag{2.7}$$

where $\mathrm{Aut}^0(E_H)$ is the connected component of the group of all automorphisms of the $H$-bundle $E_H$ containing the identity automorphism; the group $Z_0(H)$ acts on $E_H$ as translations (using the action of $H$ on $E_H$), which makes $Z_0(H)$ a subgroup of $\mathrm{Aut}^0(E_H)$.

**Theorem 2.2.** *If the reduction $E_H$ in* (2.6) *is $\Gamma$-equivariant, then the subgroup $Z_0(H) \subset \mathrm{Aut}^0(E_G)$ in* (2.7) *is contained in* $\mathrm{Aut}(E_G)^\Gamma$ *(defined in* (2.5)*).*

*Conversely, if $Z_0(H) \subset \mathrm{Aut}^0(E_G) \cap \mathrm{Aut}(E_G)^\Gamma$, then the reduction $E_H$ in* (2.6) *is $\Gamma$-equivariant.*

**Proof.** Assume that the reduction $E_H$ in (2.6) is $\Gamma$-equivariant. For any $\gamma \in \Gamma$, let $\Phi_\gamma$ be the automorphism of the variety $E_H$ defined by the action of $\gamma$. The automorphism $g \in Z_0(H) \subset \mathrm{Aut}^0(E_G)$ preserves $E_H$, and on $E_H$ it coincides with the map $z \mapsto zg$. Let $S_g$ be the automorphism of the $H$-bundle $E_H$ defined by $z \mapsto zg$. Since the actions of $G$ and $\Gamma$ on $E_G$ commute, we have,

$$\Phi_\gamma \circ S_g \circ \Phi_\gamma^{-1} = S_g \circ \Phi_\gamma \circ \Phi_\gamma^{-1} = S_g,$$

on $E_H$. Therefore, the two automorphisms, namely $g \in \mathrm{Aut}^0(E_G)$ (in (2.7)) and the image of $g$ by the action $\gamma$ on $\mathrm{Aut}(E_G)$, coincide over $E_H \subset E_G$. Consequently, these two automorphisms of $E_G$ coincide. In other words, the action of $\Gamma$ on $\mathrm{Aut}(E_G)$ fixes the subgroup $Z_0(H)$ pointwise. This completes the proof of the first part.

Assume that $\Gamma$ acts trivially on the subgroup $Z_0(H) \subset \mathrm{Aut}(E_G)$ defined in (2.7). Take a closed point $x \in M$. We will show that the evaluation map,

$$f_x : Z_0(H) \to \mathrm{Ad}(E_G)_x, \tag{2.8}$$

is injective, where $\mathrm{Ad}(E_G)_x$ is the fiber of $\mathrm{Ad}(E_G)$ over $x$; the map $f_x$ sends any $s \in Z_0(H)$ to the evaluation at $x$ of the corresponding section (as in (2.7)) of $\mathrm{Ad}(E_G)$.

To prove that the homomorphism $f_x$ is injective, fix a finite dimensional faithful left $G$-module $V$ defined over $k$. Let,

$$E_V := (E_G \times V)/G,$$

be the vector bundle over $M$ associated to $E_G$ for the $G$-module $V$; the action of any $g \in G$ sends $(z, v) \in E_G \times V$ to $(zg, g^{-1}v)$. Take any $\sigma \in Z_0(H) \subset \mathrm{Aut}^0(E_G)$. So $\sigma$ gives an automorphism,

$$\sigma' \in H^0\big(M, \mathrm{Isom}(E_V)\big)$$

of the vector bundle $E_V$; the automorphism of $E_G \times V$ that sends any $(z, v) \in E_G \times V$ to $(\sigma(z), v)$ descends to an automorphism of $E_V$.

Since $M$ is complete and irreducible, there are no nonconstant functions on it. Therefore, the coefficients of the characteristic polynomial of the endomorphism,

$$\sigma'(y) \in \operatorname{End}\big((E_V)_y\big),$$

where $y \in M$ is a closed point, are independent of $y$. Also, since $\sigma$ is an element of a torus, namely $Z_0(H)$, the endomorphism $\sigma'(y)$ is semisimple.

If $f_x(\sigma) = \mathrm{Id}_{(E_G)_x}$, where $f_x$ is defined in (2.8), then clearly $\sigma'(x) = \mathrm{Id}_{(E_V)_x}$. Therefore, in that case, all the eigenvalues of $\sigma'(y)$ are 1 for all $y \in M$. Since all $\sigma'(y)$ is semisimple with all eigenvalues 1, it follows immediately that $\sigma'(y)$ is the identity automorphism of $(E_V)_y$ for each $y \in M$.

Since $V$ is a faithful $G$-module and $\sigma'$ is the identity automorphism of $E_V$, we conclude that $\sigma$ is the identity automorphism of $E_G$. This proves that the homomorphism $f_x$ defined in (2.8) is injective.

Therefore, using the evaluation map, $M \times Z_0(H) \subset \mathrm{Ad}(E_G)$ is a subgroup-scheme. Since $Z_0(H)$ is preserved by the action of $\Gamma$ on $\mathrm{Aut}(E_G)$, it follows immediately that the action of $\Gamma$ on $\mathrm{Ad}(E_G)$ leaves this subgroup-scheme invariant.

Fix an element $g_0 \in Z_0(H)$ such that the Zariski closure of the group generated by $g_0$ coincides with $Z_0(H)$. Since $H$ is the centralizer of the subgroup $Z_0(H) \subset G$, and the algebraic subgroup generated by $g_0$ coincides with $Z_0(H)$, we conclude that $H$ coincides with the centralizer of $g_0 \in G$.

Let,

$$F : E_G \times G \to \mathrm{Ad}(E_G) := (E_G \times G)/G, \tag{2.9}$$

be the quotient map. Let,

$$\widehat{F} := F^{-1}\big(\mathrm{image}(\hat{g}_0)\big) \subset E_G \times G, \tag{2.10}$$

be the subvariety, where

$$\hat{g}_0 : M \to \mathrm{Ad}(E_G) \tag{2.11}$$

is the section defined by the above element $g_0$ using the inclusion $Z_0(H) \hookrightarrow \mathrm{Aut}(E_G)$ in (2.7). Set:

$$\widehat{E} := \widehat{F} \cap \big(E_G \times \{g_0\}\big) \subset E_G \times G, \tag{2.12}$$

where $\widehat{F}$ is defined in (2.10), and let,

$$E' \subset E_G, \tag{2.13}$$

be the image of $\widehat{E}$ (constructed in (2.12)) by the projection of $E_G \times G$ to $E_G$ defined by $(z, g) \mapsto z$.

Since $\Gamma$ acts trivially on the subgroup $Z_0(H) \hookrightarrow \mathrm{Aut}(E_G)$, the image of the map $\hat{g}_0$ in (2.11) is left invariant by the action of $\Gamma$ on $\mathrm{Ad}(E_G)$. Since the action of $\Gamma$ on $\mathrm{Ad}(E_G)$ is the descent, by the projection $F$ in (2.9), of the diagonal action on $E_G \times G$ with $\Gamma$ acting trivially on $G$, it follows that $E'$ in (2.13) is left invariant by the action of $\Gamma$ on $E_G$.

Since $E'$ is left invariant by $\Gamma$, the theorem follows once we show that $E'$ coincides with the subvariety $E_H$ in (2.6).

To prove that $E' = E_H$, first note that

$$E_H \times \{g_0\} \subset \widehat{F} \subset E_G \times G,$$

with $\widehat{F}$ defined in (2.10). Indeed, the automorphism of $E_H$ defined by $g_0$ sends any $z \in E_H$ to $zg_0$ (since $g_0$ is in the center of $H$, this commutes with the action of $H$ and hence it is an automorphism of $E_H$). This immediately implies that $E_H \times \{g_0\} \subset \widehat{F}$. Consequently, we have $E_H \subset E'$. On the other hand, for any $x \in M$ and $w \in E' \cap (E_G)_x$ it can be shown that the fiber of $E'$ over $x$ is contained in the orbit of $w$ for the action of the centralizer of $g_0$ in $G$. Indeed, if $F(w', g') = F(w'g, g')$, where $g, g' \in G$, $w' \in (E_G)_x$ and $F$ as in (2.9), then $gg'g^{-1} = g'$, this being an immediate consequence of the definition of $F$. Therefore, if $w, wg \in E'$, with $g \in G$, then $g^{-1}g_0 g = g_0$.

We already noted that the centralizer of $g_0$ in $G$ coincides with $H$. We also saw that $E_H \subset E'$. Therefore, the above observation that any two points of $E'$ over a point $x \in M$ differ by an element of the centralizer of $g_0$ implies that $E_H = E'$. This completes the proof of the theorem. □

**Example 2.3.** It may happen that $\Gamma$ preserves the subgroup $Z_0(H) \subset \mathrm{Aut}^0(E_G)$ in (2.7), but does not preserve $Z_0(H)$ pointwise. We give an example.

Fix a maximal torus $T \subset G$. Take $\Gamma$ to be the normalizer $N(T)$ of $T$ in $G$, and equip $M$ with the trivial action of $\Gamma$; let $G$ be such that $N(T) \neq T$. Set $E_G$ to be the trivial $G$-bundle $M \times G$. The group $N(T)$ acts on $M \times G$ as left translations of $G$. So the induced action of $N(T)$ on $\mathrm{Aut}(E_G) = G$ is the conjugation action. Set $H = T$. The reduction of structure group of $E_G$ to $T$ defined by the inclusion $M \times T \hookrightarrow M \times G$ has the property that the subgroup

$$Z_0(H) = T \subset G = \mathrm{Aut}(E_G)$$

(defined in (2.7)) is left invariant by the action of $N(T)$ (in this case it is the adjoint action of $N(T)$ on $G$). However no reduction of structure group of $E_G$ to $T$ is left invariant by the action $N(T)$.

The automorphism group of a torus is a discrete group. Therefore, if $\Gamma$ is a connected algebraic group acting algebraically on $E_G$, then $\Gamma$ acts trivially on $Z_0(H)$ provided $Z_0(H)$ is preserved by $\Gamma$.

**Proposition 2.4.** *Let $T \subset \mathrm{Aut}^0(E_G) \cap \mathrm{Aut}^0(E_G)^\Gamma$ be a torus such that there is an element $g \in \mathrm{Aut}^0(E_G)$ satisfying the condition that $g^{-1}Tg = Z_0(H)$, with $Z_0(H)$ constructed in* (2.7) *for the reduction $E_H$ in* (2.6)*. Then the principal $G$-bundle $E_G$ admits a $\Gamma$-equivariant reduction of structure group to the Levi subgroup $H$.*

**Proof.** Take $T$ and $g$ as above. So, the image,

$$E'_H := g(E_H) \subset E_G,$$

is a reduction of structure group of $E_G$ to $H$, where $E_H$ is the reduction in (2.6). Take any automorphism $\tau$ of the principal $H$-bundle $E_H$. Using the reduction $E_H$ in (2.6), the automorphism $\tau$ gives an automorphism $\tau_1$ of the $G$-bundle $E_G$. On the other hand, using the above reduction $E'_H \subset E_G$ together with the isomorphism of $E_H$ with $E'_H$ defined by $z \mapsto g(z)$ the automorphism $\tau$ gives an automorphism $\tau_2$ of $E_G$. It is easy to see that $\tau_2 = g\tau_1 g^{-1}$.

Therefore, if we substitute $E_H$ by $E'_H$, then the subgroup $Z_0(H) \subset \mathrm{Aut}^0(E_G)$ in (2.7) gets replaced by $gZ_0(H)g^{-1}$. Now, the second part of Theorem 2.2 says that $E'_H$ is left invariant by the action of $\Gamma$ on $E_G$. This completes the proof of the proposition. □

## 3. Levi reductions from tori in $\mathrm{Aut}(E_G)^\Gamma$

Let $T \subset \mathrm{Aut}^0(E_G)$ be a torus. From the proof of Theorem 2.2 it can be deduced that $T$ determines a torus, unique up to an inner automorphism, in $G$. This will be explained below with more details.

Fix a point $x \in M$. We saw in the proof of Theorem 2.2 that the evaluation map,

$$f_x : T \to \mathrm{Ad}(E_G)_x, \tag{3.1}$$

is injective. Since $\mathrm{Ad}(E_G) = (E_G \times G)/G$, if we fix a point $z \in (E_G)_x$, then the quotient map $F$ (defined in (2.9)) gives an isomorphism of $\{z\} \times G$ with $\mathrm{Ad}(E_G)_x$. This identification of $G$ with $\mathrm{Ad}(E_G)_x$ constructed using $z$ is an isomorphism of algebraic groups. Furthermore, if we substitute $z$ by $zg$, $g \in G$, then the corresponding isomorphism of $G$ with $\mathrm{Ad}(E_G)_x$ is the composition of the earlier one with the automorphism of $G$ defined by the conjugation action of $g$. Therefore, $f_x(T)$, with $f_x$ defined in (3.1), gives a torus in $G$ up to conjugation.

This torus of $G$, up to conjugation, defined by $f_x(T)$ actually does not depend on the choice of the point $x$. To prove this, take $z_1, z_2 \in E_G$ with $f(z_i) = x_i$, $i = 1, 2$, where $f$ is as in (2.2). Consider the evaluation homomorphism:

$$f_{x_i} : T \to \mathrm{Ad}(E_G)_{x_i}$$

which is injective. Let,

$$h_{z_i} : T \to G, \tag{3.2}$$

be the composition of $f_{x_i}$ with the identification of $\mathrm{Ad}(E_G)_{x_i}$ with $G$ defined by $z_i$. We want to show that the two subgroups, namely $\mathrm{image}(h_{z_1})$ and $\mathrm{image}(h_{z_2})$, of $G$ differ by an inner automorphism of $G$.

Fix a point $t_0 \in T$ such that the Zariski closed subgroup of $T$ generated by $t_0$ is $T$ itself. For a finite dimensional left $G$-module $V$ defined over $k$, let $E_V$ be the vector bundle associated to $E_G$ for $V$ and $\hat{t}_0$ the automorphism of $E_V$ defined by $t_0 \in \mathrm{Aut}(E_G)$. From the definition of the map $h_{z_i}$ it follows that the automorphism $\hat{t}_0(x_i)$ of $(E_V)_{x_i}$

and the automorphism of $V$ given by $h_{z_i}(t_0) \in G$ are intertwined by the isomorphism of $(E_V)_{x_i}$ with $V$ constructed using $z_i$. (Since $E_V = (E_G \times V)/G$, we have an isomorphism of $(E_V)_{x_i}$ with $V$ that sends any $v \in V$ to the image of $(z_i, v)$.) We saw in the proof of Theorem 2.2 that the characteristic polynomial of $\hat{t}_0(y) \in \mathrm{Isom}((E_V)_y)$ is independent of $y$. Therefore, the automorphisms of $V$ defined the two elements $h_{z_1}(t_0)$ and $h_{z_2}(t_0)$ of $G$ have same characteristic polynomial.

On the other hand, if $T'' \subset G$ is a maximal torus, then the algebra of all functions on the affine variety $T''/W$, where $W := N(T'')/T''$ is the Weyl group with $N(T'')$ the normalizer of $T''$ in $G$, is generated by trace function of finite dimensional left $G$-modules defined over $k$ [10, p. 87, Theorem 2]. Therefore, $h_{z_1}(t_0)$ and $h_{z_2}(t_0)$ differ by an inner automorphism of $G$ (since the characteristic polynomials of $h_{z_1}(t_0)$ and $h_{z_2}(t_0)$ coincide for any $G$-module). Since image$(h_{z_i})$ is generated, as a Zariski closed subgroup, by $h_{z_i}(t_0)$, we conclude that the two subgroups image$(h_{z_1})$ and image$(h_{z_2})$ differ by an inner automorphism of $G$.

**Remark 3.1.** Let $E_H \subset E_G$ be a reduction of structure group to a Levi subgroup $H \subset G$. Consider the torus $Z_0(H) \subset \mathrm{Aut}^0(E_G)$ in (2.7) corresponding to the reduction $E_H$. By substituting a point of $E_H$ for the point $z_i$ in (3.2) we conclude that the map in (3.2) sends any $g \in Z_0(H) \subset H$ to the point $g \in \mathrm{Aut}^0(E_G)$ (in terms of (2.7)). Consequently, the torus $Z_0(H) \subset G$ is in the conjugacy class of tori given by the torus $Z_0(H) \subset \mathrm{Aut}^0(E_G)$ in (2.7).

We have the following lemma:

**Lemma 3.2.** *If the $G$-bundle $E_G$ admits a $\Gamma$-equivariant reduction of structure group to a Levi subgroup $H \subset G$, then there is a torus $T \subset \mathrm{Aut}^0(E_G) \cap \mathrm{Aut}(E_G)^\Gamma$ that satisfies the condition that $Z_0(H)$ is the torus in $G$ defined, up to conjugation, by $T$.*

**Proof.** Let $E_H \subset E_G$ be a $\Gamma$-equivariant reduction of structure group to $H$. The image of $Z_0(H)$ in $\mathrm{Aut}^0(G)$ by the inclusion map in (2.7) will be denoted by $T$. The first part of Theorem 2.2 says that $T \subset \mathrm{Aut}(E_G)^\Gamma$.

Fix a point $z \in E_H \subset E_G$. It is easy to see that the torus $h_z(T) \subset G$ coincides with $Z_0(H)$, where $h_z$ is defined as in (3.2) (by composing the evaluation map $T \to \mathrm{Ad}(E_G)_{f(z)}$, where $f$ is defined in (2.2), with the isomorphism $\mathrm{Ad}(E_G)_{f(z)} \to G$ defined by $z$). This completes the proof of the lemma. □

In the converse direction we have:

**Proposition 3.3.** *Let $T' \subset G$ be a torus in the conjugacy class of tori determined by a torus $T \subset \mathrm{Aut}^0(E_G)^\Gamma$ and $H$ the centralizer of $T'$ in $G$. Then $E_G$ admits a $\Gamma$-equivariant reduction of structure group to the Levi subgroup $H$.*

**Proof.** Fix any point $z \in E_G$ and consider the homomorphism

$$h_z : T \to G \tag{3.3}$$

as in (3.2), namely it is the composition of the evaluation map with the identification, constructed using $z$, of $G$ with $\mathrm{Ad}(E_G)_{f(z)}$, where $f$ is defined in (2.2). There is an element $g \in G$ with $gh_z(T)g^{-1} = T'$, where $T'$ is as in the statement of the proposition.

Let,

$$H_z \subset G, \tag{3.4}$$

be the centralizer of $h_z(T)$, with $h_z$ defined in (3.3). Since $gh_z(T)g^{-1} = T'$, and the centralizer of $T' \subset G$ is $H$, we conclude that

$$gH_zg^{-1} = H. \tag{3.5}$$

Fix an element $t_0 \in T$ such that the Zariski closure in $T$ of the subgroup generated by $t_0$ is $T$ itself. As in (2.11), let

$$\hat{t}_0 : M \to \mathrm{Ad}(E_G)$$

be the section defined by the automorphism $t_0 \in \mathrm{Aut}(E_G)$. As in (2.10), set:

$$\widehat{F} := F^{-1}\big(\mathrm{image}(\hat{t}_0)\big) \subset E_G \times G,$$

where $F$ is the projection in (2.9). As in (2.12), define:

$$\widehat{E} := \widehat{F} \cap \big(E_G \times \{h_z(t_0)\}\big) \subset E_G \times G,$$

where $h_z$ is defined in (3.3). Let,

$$E' \subset E_G, \tag{3.6}$$

be the image of $\widehat{E}$ by the projection of $E_G \times G$ to $E_G$ defined by $(y, \nu) \mapsto y$.

We will show that $E'$ constructed in (3.6) is a $\Gamma$-equivariant reduction of structure group of $E_G$ to the subgroup $H_z$ defined in (3.4).

For this, we will first show that $E'$ is closed under the action of $H_z$ (for the action of $G$ on $E_G$). Note that $h_z(t_0)$ is in the center of $H_z$, this being a consequence of the fact that $H_z$ is the centralizer of $h_z(T)$. Therefore, for the projection $F$ in (2.9), we have:

$$F\big(z_1, h_z(t_0)\big) = F\big(z_1 g_1, h_z(t_0)\big) \tag{3.7}$$

for all $z_1 \in E_G$ and $g_1 \in H_z$. Indeed, the map $F$ clearly has the property that for $g, g' \in G$ and $w' \in E_G$,

$$F(w', g') = F(w'g, g'), \tag{3.8}$$

if and only if $gg'g^{-1} = g'$. From (3.7) it follows immediately that $E'$ is closed under the action of $H_z$.

It also follows from (3.8) that for any point $y \in M$, the centralizer of $h_z(t_0)$ (in $G$) acts transitively on the fiber of $E'$ over $y$. Note that since the Zariski closure of the group generated by $t_0$ is $T$, and $H_z$ is the centralizer (in $G$) of $h_z(T)$, it follows immediately that the centralizer of $h_z(t_0)$ is $H_z$.

We still need to show that the fiber of $E'$ over each point $y \in M$ is nonempty. For this note that there is a point $z' \in f^{-1}(y)$, with $f$ defined in (2.2), such that the corresponding homomorphism,

$$h_{z'} : T \to G,$$

defined as in (3.3) by replacing $z$ by $z'$ has the property that $h_{z'}(t_0) = h_z(t_0)$. Indeed, this follows from the combination of the fact that the conjugacy class of the torus $h_z(T) \subset G$ is independent of the choice of the point $z \in E_G$ and the observation that the two homomorphisms $h_{z_1}$ and $h_{z_1 g_1}$ from $T$ to $G$, where $z_1 \in E_G$ and $g_1 \in G$, differ by the inner automorphism of $G$ defined by $g_1$. The identity $h_{z'}(t_0) = h_z(t_0)$ immediately implies that $z'$ is in the fiber of $E'$ over $y$.

Consequently, $E' \subset E_G$ constructed in (3.6) is a reduction of structure group to $H_z$.

Since the action of $\Gamma$ on $\mathrm{Aut}(E_G)$ fixes $t_0$, the action of $\Gamma$ on $E_G$ leaves $E'$ invariant.

Finally, from (3.5) it follows immediately that $E'g^{-1} \subset E_G$ is a reduction of structure group to $H$. As $E'$ is left invariant by the action of $\Gamma$ on $E_G$, and the actions of $\Gamma$ and $G$ on $E_G$ commute, the subvariety $E'g^{-1} \subset E_G$ is also left invariant by the action of $\Gamma$. This completes the proof of the proposition. □

## 4. A canonical equivariant Levi reduction

Let $T \subset \mathrm{Aut}(E_G)^\Gamma$ be a connected maximal torus, where $\mathrm{Aut}(E_G)^\Gamma$ is defined in (2.5). So $T$ is a torus of $\mathrm{Aut}^0(E_G)$. We saw in the previous section that $T$ determines a torus, unique up to an inner conjugation, in $G$. We will show that this torus in $G$ (up to conjugation) does not depend on the choice of the maximal torus $T$.

To prove this, first note that any two maximal tori of $\mathrm{Aut}(E_G)^\Gamma$ differ by an inner automorphism of $\mathrm{Aut}(E_G)^\Gamma$. Consider the maximal torus $g_0 T g_0^{-1}$, where $g_0 \in \mathrm{Aut}(E_G)^\Gamma$, and fix a point $z \in E_G$. The point $z$ defines two injective homomorphisms,

$$h_z : T \to G,$$

and

$$h'_z : g_0 T g_0^{-1} \to G,$$

defined as in (3.2) using the evaluation map and the isomorphism of algebraic groups,

$$\phi_z : \mathrm{Ad}(E_G)_{f(z)} \to G,$$

constructed using $z$, where $f$ is defined in (2.2). From the construction of $h_z$ and $h'_z$ it follows immediately that

$$\phi_z\big(g_0\big(f(z)\big)\big)h_z(T)\big(\phi_z\big(g_0\big(f(z)\big)\big)\big)^{-1} = h'_z\big(g_0 T g_0^{-1}\big).$$

Therefore, $h_z(T)$ and $h'_z(g_0 T g_0^{-1})$ differ by an inner automorphism of $G$. Consequently, the torus of $G$ determined by a maximal torus of $\mathrm{Aut}(E_G)^\Gamma$ does not depend on the choice of the maximal torus.

Fix a torus $T_0 \subset G$ in the conjugacy class of tori given by a maximal torus in $\mathrm{Aut}(E_G)^\Gamma$. The centralizer of $T_0$ in $G$ is a Levi subgroup. This Levi subgroup of $G$ will be denoted by $H_0$.

**Theorem 4.1.** *The principal $G$-bundle $E_G$ admits a $\Gamma$-equivariant reduction of structure group to the Levi subgroup $H_0$ defined above.*

*If $H \subsetneq H_0$ is a Levi subgroup of $G$ properly contained in $H_0$, then $E_G$ does not admit any $\Gamma$-equivariant reduction of structure group to $H$.*

*If $H \subset G$ is a Levi subgroup such that $E_G$ admits a $\Gamma$-equivariant reduction of structure group to $H$, but $E_G$ does not admit a $\Gamma$-equivariant reduction of structure group to any Levi subgroup properly contained in $H$, then $H$ is conjugate to the above defined subgroup $H_0 \subset G$.*

*If $E_{H_0} \subset E_G$ and $E'_{H_0} \subset E_G$ are two $\Gamma$-equivariant reductions of structure group to $H_0$, then there is an automorphism $\tau \in \mathrm{Aut}(E_G)^\Gamma$ of $E_G$ such that $\tau(E_{H_0}) = E'_{H_0} \subset E_G$.*

**Proof.** That $E_G$ admits a $\Gamma$-equivariant reduction of structure group to $H_0$ follows from the construction in Proposition 3.3. Fix a maximal torus $T \subset \mathrm{Aut}^0(E_G)^\Gamma$ and a point $t_0 \in T$ such that the Zariski closure of the subgroup of $T$ generated by $t_0$ coincides with $T$. Let $t'_0 \in T_0$ be the element corresponding to $t_0$ by an isomorphism of $T$ with $T_0$ constructed using an element of $E_G$. As in (2.12), consider:

$$\widehat{E} := F^{-1}\big(\mathrm{image}(\hat{t}_0)\big) \cap \big(E_G \times \{t'_0\}\big) \subset E_G \times G,$$

where $F$ is defined in (2.9) and $\hat{t}_0$ is the section of $\mathrm{Ad}(E_G)$ defined by $t_0$. Finally the image of $\widehat{E}$ by the projection of $E_G \times G$ to $E_G$ gives a reduction of structure group of $E_G$ to $H_0$. See the proof of Proposition 3.3 for the details.

To prove the second statement, let $H \subsetneq H_0$ be a Levi subgroup of $G$ properly contained in $H_0$. So $\dim Z_0(H) > \dim T_0$, where $Z_0(H)$ is the connected component of the center of $H$ containing the identity element (note that $T_0$ is contained in the center of the bigger Levi subgroup). The first statement in Theorem 2.2 says that if $E_H \subset E_G$ is a $\Gamma$-equivariant reduction of structure group of $E_G$ to $H$, then $\mathrm{Aut}(E_G)^\Gamma$ contains a torus isomorphic to $Z_0(H)$. This is impossible, since a smaller dimensional torus, namely $T_0$, is isomorphic to the maximal torus $T$ and any two maximal tori are isomorphic.

Let $H \subset G$ be a Levi subgroup as in the third statement, and let $E_H \subset E_G$ be a $\Gamma$-equivariant reduction of structure group to $H$. The condition on $H$ implies that the torus $Z_0(H) \subset \mathrm{Aut}(E_G)^\Gamma$ in (2.7) for the reduction $E_H$ is a maximal torus of $\mathrm{Aut}(E_G)^\Gamma$. Indeed, that $Z_0(H) \subset \mathrm{Aut}(E_G)^\Gamma$ follows from Theorem 2.2. That $Z_0(H)$ is a maximal torus of $\mathrm{Aut}(E_G)^\Gamma$ can be seen as follows. If $T'' \subset \mathrm{Aut}(E_G)^\Gamma$ is a torus with $Z_0(H) \subsetneq T''$, then take a torus $T''_1$ in the conjugacy class of tori of $G$ given by $T''$ such that $Z_0(H) \subset T''_1 \subset G$. Let $H'' \subset G$ be the centralizer of $T''_1$. Since $Z_0(H)$ is the connected component of the center of $H$ containing the identity element and $Z_0(H) \subsetneq T''_1$ is a proper subtorus, we conclude that $H'' \subsetneq H$. Proposition 3.3 says that $E_G$ admits a $\Gamma$-equivariant reduction of structure group to $H''$. Since $H''$ is a Levi subgroup properly contained in $H$, this contradicts the given condition on $H$. Therefore, $Z_0(H) \subset \mathrm{Aut}(E_G)^\Gamma$ is a maximal torus.

Since $T_0$, by definition, is in the conjugacy class of tori of $G$ given by a maximal torus of $\mathrm{Aut}(E_G)^\Gamma$, using Remark 3.1 we conclude that the two tori $T_0$ and $Z_0(H)$ of $G$ are conjugate. Consequently, $H$ and $H_0$ differ by an inner automorphism of $G$.

Let $E_{H_0}$ and $E'_{H_0}$ be as in the fourth statement. Consider the inclusion in (2.7). Let $T_1$ (respectively, $T'_1$) be the image of $T_0$ in $\mathrm{Aut}(E_G)^\Gamma$ for the reduction $E_{H_0}$ (respectively, $E'_{H_0}$) by (2.7). From dimension consideration we know that both $T_1$ and $T'_1$ are maximal tori in $\mathrm{Aut}(E_G)^\Gamma$. Take an element $\tau \in \mathrm{Aut}(E_G)^\Gamma$ such that

$$T'_1 = \tau^{-1} T_1 \tau. \tag{4.1}$$

Let $E_H \subset E_G$ be a $\Gamma$-equivariant reduction of structure group to a Levi subgroup $H \subset G$ and $g_0 \in Z_0(H)$ an element in the connected component of the center of $H$ containing the identity element such that $g_0$ generates $Z_0(H)$

as a Zariski closed subgroup of $G$. In the proof of Theorem 2.2 we gave a reconstruction of $E_H$ from $g_0$ and its image in $\mathrm{Aut}^0(E_G)$ by (2.7). (In the notation of the proof of Theorem 2.2, $E' \subset E_G$ was constructed in (2.13) using $g_0$ and its image in $\mathrm{Aut}^0(E_G)$, and it was shown there that $E_H$ coincides with $E'$.)

Fix an element $g_0 \in T_0$ such that Zariski closed subgroup generated by $g_0$ coincides with $T_0$. Let $g_1$ be the image of $g_0$ in $T_1$ for the above isomorphism of $T_0$ with $T_1$ constructed using $E_H$. Set,

$$g_1' = \tau^{-1} g_1 \tau \in T_1',$$

where $\tau$ is as in (4.1). Let $g_0'$ be the image of $g_1'$ in $T_0$ for the above isomorphism of $T_1'$ with $T_0$ constructed using $E_H'$. Following the construction of $E'$ in (2.13), we can reconstruct $E_H$ (respectively, $E_H'$) using the pair $(g_0, g_1)$ (respectively, $(g_0', g_1')$). Using this reconstruction it is straightforward to see that

$$E_H' = \tau^{-1}(E_H),$$

where $\tau$ is as in (4.1). This completes the proof of the theorem. □

**Remark 4.2.** If we set $G = \mathrm{GL}(n,k)$ and $\Gamma = \{e\}$, then Theorem 4.1 becomes the following theorem proved in [1]: any vector bundle $V$ over $M$ is isomorphic to a direct sum $\bigoplus_{i=1}^{\ell_0} U_i$ of indecomposable vector bundles, and if

$$V \cong \bigoplus_{j=1}^{\ell} W_j,$$

where each $W_j$ is indecomposable, then $\ell_0 = \ell$ and the collection of vector bundles $\{W_j\}$ is a permutation of $\{U_i\}$.

**Remark 4.3.** Let $E_*$ be a parabolic $G$-bundle over an irreducible smooth projective variety $X$. Corresponding to $E_*$, there is an irreducible smooth projective variety $Y$, a finite subgroup $\Gamma \subset \mathrm{Aut}(Y)$ with $X = Y/\Gamma$, and a principal $G$-bundle $E_G$ over $Y$ equipped with a lift of the action of $\Gamma$. More precisely, there is a bijective correspondence between parabolic $G$-bundles and $G$-bundles with a finite group action on a (ramified) covering (see [2] for the details). Therefore, Theorem 4.1 gives a natural reduction of structure group of a parabolic $G$-bundle to a Levi subgroup of $G$. This Levi reduction satisfies all the analogous conditions in Theorem 4.1.

In the rest of this section we will give four examples where Theorem 4.1 apply.

**Example 4.4.** For the first example, take $E_G$ to be the trivial $G$-bundle $M \times G$ over $M$. Let $H' \subset G$ be any subgroup. Set $\Gamma = H'$, with $H'$ acting on $M \times G$ as left-translations on $G$. So the action of $\Gamma = H'$ on $M$ is the trivial action. Let $H$ be the smallest Levi subgroup of $G$ containing $H'$. Let $H'' \subset G$ be the centralizer of $H'$. The Levi subgroup $H$ is the centralizer of the unique maximal torus of the center of $H''$. It is easy to see that the subgroup $H_0$ in Theorem 4.1 for this example is $H$ (up to an inner conjugation), and $E_{H_0} = M \times H \subset M \times G =: E_G$.

**Example 4.5.** For the second example, take $G$ to be a simple linear algebraic group defined over $k$. Fix a proper parabolic subgroup $P \subsetneq G$. Also, fix a Levi subgroup $H \subset P$. So, the quotient $P/R_u(P)$, where $R_u(P)$ is the unipotent radical of $P$, is identified with $H$ by the quotient morphism $P \to P/R_u(P)$. Let,

$$\rho : P \to H, \tag{4.2}$$

be the projection defined by the quotient morphism $P \to P/R_u(P)$ and the identification of $H$ with $P/R_u(P)$ given by it.

Set $M = G/P$. Let $E_H$ be the principal $H$-bundle over $G/P$ defined by the quotient:

$$E_H := (G \times H)/P, \tag{4.3}$$

where the action of any $h \in P$ sends any $(g, g') \in G \times H$ to $(gh, \rho(h)^{-1}g')$; the homomorphism $\rho$ is defined in (4.2). The quotient morphism $P \to G/P$ defines a principal $P$-bundle over $G/P$. The principal $H$-bundle obtained by extending the structure group of this principal $P$-bundle using the projection $\rho$ in (4.2) is identified with the principal $H$-bundle $E_H$ defined in (4.3). Let $E_G$ be the principal $G$-bundle over $G/P$ obtained by extending the structure group of this principal $G$-bundle $E_H$ using the inclusion of $H$ in $G$. Therefore, we have an identification:

$$E_G = (G \times G)/P, \tag{4.4}$$

where the action of any $h \in P$ sends any $(g, g') \in G \times G$ to $(gh, \rho(h)^{-1}g')$; the homomorphism $\rho$ is defined in (4.2).

Set $\Gamma$ to be any subgroup of $G$; both the cases $\Gamma = G$ and $\Gamma = \{e\}$ are allowed. The group $\Gamma$ acts on $G/P$ through the left-translation action of $G$ on $G/P$. The group $\Gamma$ acts on $E_G$ as the left-translation action of $G$ on the left-hand side factor $G$ in $G \times G$ in terms of the identification in (4.4). Clearly, the subvariety,

$$E_H = (G \times H)/P \subset (G \times G)/P = E_G, \tag{4.5}$$

where $E_H$ is defined in (4.3), is left invariant by the action of $\Gamma$ on $E_G$. The principal $H$-bundle $E_H$ is an example of a homogeneous bundle over $G/P$.

Our aim is to show that the subgroup $H_0$ in Theorem 4.1 for this example is $H$ (up to an inner conjugation), and $E_{H_0}$ is the principal $H$-bundle $E_H$ in (4.5). To prove this it suffices to show that the principal $H$-bundle $E_H$ in (4.5) does not admit any $\Gamma$-equivariant reduction of structure group to any Levi subgroup of $G$ properly contained in $H$ (see the third and the fourth statements in Theorem 4.1).

Let $\mathfrak{h}$ denote the Lie algebra of $H$. The center of $\mathfrak{h}$ will be denoted by $\mathfrak{z}(\mathfrak{h})$. Let,

$$\mathrm{ad}(E_H) = (E_H \times \mathfrak{h})/H,$$

be the adjoint vector bundle of the principal $H$-bundle $E_H$ over $G/P$. Since $H$ acts trivially on the subalgebra $\mathfrak{z}(\mathfrak{h})$ of $\mathfrak{h}$, the trivial vector bundle $(G/P) \times \mathfrak{z}(\mathfrak{h})$ is a subbundle of $\mathrm{ad}(E_H)$. Consequently, we have:

$$\mathfrak{z}(\mathfrak{h}) \subset H^0\big(G/P, \mathrm{ad}(E_H)\big). \tag{4.6}$$

Let $E_{H'} \subset E_H$ be a $\Gamma$-equivariant reduction of structure group to a Levi subgroup $H' \subset G$ properly contained in $H$. Then the dimension of the center $\mathfrak{z}(\mathfrak{h}')$ of the Lie algebra $\mathfrak{h}'$ of $H'$ is strictly greater than the dimension of $\mathfrak{z}(\mathfrak{h})$. Also, as in (4.6), we have:

$$\mathfrak{z}(\mathfrak{h}') \subset H^0\big(G/P, \mathrm{ad}(E_{H'})\big) \subset H^0\big(G/P, \mathrm{ad}(E_H)\big),$$

where $\mathrm{ad}(E_{H'})$ is the adjoint bundle of the principal $H'$-bundle $E_{H'}$. Hence it follows that

$$\dim \mathfrak{z}(\mathfrak{h}) < \dim \mathfrak{z}(\mathfrak{h}') \leqslant \dim H^0\big(G/P, \mathrm{ad}(E_H)\big),$$

or equivalently, the inclusion in (4.6) is a proper inclusion.

Therefore, to prove that the principal $H$-bundle $E_H$ in (4.5) does not admit any $\Gamma$-equivariant reduction of structure group to any Levi subgroup of $G$ properly contained in $H$ it is enough to show that the inclusion in (4.6) is actually an isomorphism.

The following proposition, which says that the inclusion in (4.6) is actually an isomorphism, completes the proof of the assertion that the subgroup $H_0$ in Theorem 4.1 for this second example is $H$ (up to an inner conjugation), and $E_{H_0}$ (in Theorem 4.1) is the principal $H$-bundle $E_H$ defined in (4.3).

**Proposition 4.6.** *The inclusion $\mathfrak{z}(\mathfrak{h}) \subset H^0(G/P, \mathrm{ad}(E_H))$ in (4.6) is an isomorphism.*

**Proof.** Consider the Lie algebra $\mathfrak{h}$ as a $H$-module using the adjoint action. Since $H$ is reductive, we have a decomposition of the $H$-module:

$$\mathfrak{h} \cong \mathfrak{z}(\mathfrak{h}) \oplus \left(\bigoplus_{i=1}^{\ell} V_i\right),$$

where $V_i$ are the simple factors of the Lie algebra of the reductive group $H$. This decomposition gives a decomposition of the adjoint vector bundle:

$$\mathrm{ad}(E_H) \cong \big((G/P) \times \mathfrak{z}(\mathfrak{h})\big) \oplus \left(\bigoplus_{i=1}^{\ell} E_{V_i}\right),$$

where $E_{V_i} = (E_H \times V_i)/H$ is the vector bundle over $G/P$ associated to the principal $H$-bundle $E_H$ for the $H$-module $V_i$; the Cartesian product $(G/P) \times \mathfrak{z}(\mathfrak{h})$ is the trivial vector bundle over $G/P$ with fiber $\mathfrak{z}(\mathfrak{h})$.

In view of the above decomposition of the adjoint vector bundle $\mathrm{ad}(E_H)$, we conclude that

$$\mathfrak{z}(\mathfrak{h}) = H^0\big(G/P, \mathrm{ad}(E_H)\big)$$

if and only if

$$H^0(G/P, E_{V_i}) = 0 \tag{4.7}$$

for all $i \in [1, \ell]$.

We will prove (4.7). For that, consider the action of $G$ on $E_H$ lifting the action left-translation action of $G$ on $G/P$. We showed that $\Gamma$ acts on $E_H$; this action of $G$ on $E_H$ coincides with the action of $\Gamma$ of $E_H$ when $\Gamma = G$. The action of $G$ on $E_H$ induces an action of $G$ on the associated vector bundle $E_{V_i}$ (recall that $E_{V_i}$ is the vector bundle associated to $E_H$ for the $G$-module $V_i$).

Take any $i \in [1, \ell]$. We assume that $H^0(G/P, E_{V_i}) \neq 0$. Take any nonzero section $\phi$ of $E_{V_i}$. We have noted above that $G$ acts on $E_{V_i}$. So $G$ acts on $H^0(G/P, E_{V_i})$. For any element $g \in G$, the translation of $\phi$ by $g$, which we will denote by $g \circ \phi$, is also a nonzero section of $E_{V_i}$. Let,

$$W_i \subset E_{V_i},$$

be the coherent subsheaf generated by all the sections $\{g \circ \phi\}_{g \in G}$. From this construction of $W_i$ it follows immediately that the action of $G$ on $E_{V_i}$ leaves this subsheaf $W_i$ invariant. Since the action of $G$ on $G/P$ is transitive, we conclude that $W_i$ is actually a subbundle of $E_{V_i}$.

The fiber of $E_{V_i}$ over the point $eP \in G/P$ is naturally identified with $V_i$. This isomorphism of $(E_{V_i})_{eP}$ with $V_i$ is defined by sending any $v \in V_i$ to the element in $(E_V)_{eP}$ defined by $(e, v)$; we recall that $E_{V_i}$ is the quotient of $G \times V_i$ by the action of $P$ (the action of any $z \in P$ sends any $(g, v) \in G \times V_i$ to $(gz, \rho(z)^{-1}(v))$, where $\rho$ is the homomorphism in (4.2)).

The isotropy subgroup $P$ of the point $eP \in G/P$ (for the action of $G$ on $G/P$) acts on the fiber $(E_{V_i})_{eP}$. The action of $P$ on $(E_{V_i})_{eP}$ factors through the quotient $H$ in (4.2) (this follows immediately from the above action of $P$ on $G \times V_i$ defining $E_{V_i}$), and the induced action of $H$ on $(E_{V_i})_{eP}$ is taken by the above identification $(E_{V_i})_{eP} = V_i$ to the action of $H$ on the $H$-module $V_i$.

We noted above that the action of $G$ on $E_{V_i}$ leaves the subbundle $W_i$ invariant. This, together with the fact that $H$ is contained in the isotropy subgroup $P$ of the point $eP \in G/P$ (for the action of $G$ on $G/P$) imply that the subspace,

$$(W_i)_{eP} \subset (E_{V_i})_{eP} = V_i,$$

is left invariant by the action of $H$ on $V_i$. On the other hand, since $V_i$ is a simple factor of the Lie algebra of th reductive group $H$, we conclude that the $H$-module $V_i$ is irreducible. Since $(W_i)_{eP}$ is a nonzero submodule of the irreducible $H$-module $V_i$, we conclude that $(W_i)_{eP} = V_i$. Therefore, we have $W_i = E_{V_i}$. In other words, the vector bundle $E_{V_i}$ is globally generated (generated by its global sections).

Let $r_i$ denote the rank of the vector bundle $E_{V_i}$. Fix $r_i$ sections,

$$\{s_1, \ldots, s_{r_i}\} \in H^0(G/P, E_{V_i})^{\oplus r_i},$$

such that for the general point $x \in G/P$, the evaluations $\{s_1(x), \ldots, s_{r_i}(x)\}$ together generate the fiber $(E_{V_i})_x$. It is easy to see that if $\{s_1(y, \ldots, s_{r_i}(y)\}$ are linearly independent for one point $y \in G/P$, then they are linearly independent over the general point. Consider the homomorphism of vector bundles,

$$F : (G/P) \times k^{\oplus r_i} \to E_{V_i}, \tag{4.8}$$

defined by $(z; \lambda_1, \ldots, \lambda_{r_i}) \mapsto \sum_{j=1}^{r_i} \lambda_j s_j(z)$, where $z \in G/P$ and $\lambda_j \in k$; here $(G/P) \times k^{\oplus r_i}$ is the trivial vector bundle over $G/P$ with fiber $k^{\oplus r_i}$. Let,

$$\bigwedge^{r_i} F : \bigwedge^{r_i}\big((G/P) \times k^{\oplus r_i}\big) = (G/P) \times k \to \bigwedge^{r_i} E_{V_i}, \tag{4.9}$$

be the homomorphism of the determinant line bundles (top exterior power of vector bundles) induced by $F$.

This homomorphism of vector bundles $F$ in (4.8) fails to be an isomorphism over a point $z \in G/P$ if and only if the homomorphism,

$$\bigwedge^{r_i} F(z) : k \to \bigwedge^{r_i} (E_{V_i})_z,$$

vanishes, where $\bigwedge^{r_i} F$ is defined in (4.9).

Since the $H$-module $V_i$ is the Lie algebra of a simple factor of the reductive group $H$, the $H$-module $V_i$ is self-dual, that is, $V_i$ is isomorphic to $V_i^*$. Consequently, the vector bundle $E_{V_i}$, which is associated to the principal $H$-bundle $E_H$ for the $H$-module $V_i$, is isomorphic to the vector bundle $(E_{V_i})^*$, which is associated to $E_H$ for the $H$-module $V_i^*$. Since $E_{V_i}$ is isomorphic to its dual $(E_{V_i})^*$, it follows immediately that the determinant line bundle $\bigwedge^{r_i} E_{V_i}$ is isomorphic to the trivial line bundle over $G/P$. Consequently, the homomorphism $\bigwedge^{r_i} F$ in (4.9), which is a nonzero homomorphism between trivializable line bundles, must be an isomorphism of line bundles.

This implies that the homomorphism $\bigwedge^{r_i} F$ vanishes nowhere. Consequently, $F$ in (4.8) is an isomorphism of vector bundles. In other words, the vector bundle $E_{V_i}$ is trivializable.

The action of $G$ on the vector bundle $E_{V_i}$ gives an action of $G$ on $H^0(G/P, E_{V_i})$. Since the vector bundle $E_{V_i}$ is isomorphic to the trivial vector bundle of rank $r_i$ (the homomorphism $F$ in (4.8) gives a trivialization of $E_{V_i}$), the vector bundle $E_{V_i}$ is canonically isomorphic to the trivial vector bundle over $G/P$ with fiber $H^0(G/P, E_{V_i})$; the isomorphism is given by the evaluation map of sections. Therefore, $H^0(G/P, E_{V_i})$ is identified with

$$(E_{V_i})_{eP} = V_i$$

(we saw above that the fiber $(E_{V_i})_{eP}$ is canonically identified with $V_i$). We noted above that $G$ acts on $H^0(G/P, E_{V_i})$. The action of the group $H$ on $H^0(G/P, E_{V_i})$ obtained by restricting the action of $G$ on $H^0(G/P, E_{V_i})$ clearly coincides with the action of $H$ on $V_i$ by the above isomorphism of $H^0(G/P, E_{V_i})$ with $V_i$.

Therefore, the action of $H$ on $V_i$ extends to an action of $G$ on $V_i$. Since $H$ is a proper Levi subgroup of the simple group $G$, using the fact that the action of $H$ on the nonzero $H$-module $V_i$ is the restriction of an action of $G$ on $V_i$ it is easy to deduce that the $H$-module $V_i$ is not irreducible. On the other hand, we noted earlier that $V_i$ is an irreducible $H$-module. Therefore, we conclude that $H^0(G/P, E_{V_i}) = 0$. This completes the proof of the proposition. □

**Example 4.7.** For the third example, again take a proper parabolic subgroup $P$ of a simple group $G$. Set $M = G/P$ and $\Gamma = G$, with $G$ acting on $G/P$ as left-translations. Set,

$$E_G = (G \times G)/P,$$

where the action of any $z \in P$ sends any $(g, h) \in G \times G$ to $(gz, z^{-1}h)$. It should be pointed out that this $G$-bundle $E_G$ is different from the one defined in (4.4); in (4.4), the action of $P$ on the right-hand side factor $G$ of $G \times G$ factors through $\rho$. The action of $G$ on $G/P$ lifts to an action of $G$ on $E_G$; the lifted action coincides with the one given by the left-translation action of $G$ on the left-hand side factor $G$ in $G \times G$.

It is easy to see that $E_G$ does not admit any $\Gamma$-equivariant reduction of structure group to any proper Levi subgroup of $G$. To prove this first note that the left translation action of $P$ on $G$ does not leave invariant any proper Levi subgroup of $G$. Now consider the action of the isotropy group $P$ of the point $eP \in G/P$ on the fiber of $E_G$ over $eP$. This fiber is canonically identified with $G$ (by sending any point $g \in G$ to the point in $(E_G)_{eP}$ defined by $(e, g)$), and the action of $P$ on $eP$ is the left-translation action of $P$ on $G$. Hence $E_G$ does not admit any $G$-equivariant reduction of structure group to any proper Levi subgroup of $G$. Therefore, $H_0 = G$ and $E_{H_0} = E_G$ in this example (see the notation in Theorem 4.1).

**Example 4.8.** For the fourth example, take the base field $k$ to be the field of complex numbers. Let $G$ be any connected reductive linear algebraic group defined over $\mathbb{C}$. Fix a Levi subgroup $H \subset G$. Let $K(G) \subset G$ be a maximal compact subgroup. So,

$$K(H) := H \cap K(G) \subset H,$$

is a maximal compact subgroup of the Levi subgroup $H$. Take a connected complex projective manifold $M$. Assume that there is a homomorphism,

$$\rho : \pi_1(M, x_0) \to K(H),$$

such that the image of $\rho$ is a dense subgroup of the topological group $K(H)$. For example, if $M$ is a compact Riemann surface of genus at least two, then there is such a homomorphism. Fix such a homomorphism $\rho$.

The homomorphism $\rho$ defines a polystable principal $G$-bundle $E_G$ over $M$ (see [8, p. 24, Theorem 1]). Let $\widetilde{M}$ be the universal cover $M$ for the base point $x_0$. Consider the quotient:

$$E_G := \big(\widetilde{M} \times G\big)/\pi_1(M, x_0),$$

where the action of any $\gamma \in \pi_1(M, x_0)$ sends any point $(x, g) \in \widetilde{M} \times G$ to $(x\gamma, \rho(\gamma)^{-1}g)$ (we identify $\pi_1(M, x_0)$ with the group of deck transformations of the universal cover). This quotient $E_G$ is a flat principal $G$-bundle over $M$, and the underlying holomorphic $G$-bundle is polystable.

Define:

$$E_H := \big(\widetilde{M} \times H\big)/\pi_1(M, x_0) \subset \big(\widetilde{M} \times G\big)/\pi_1(M, x_0) = E_G, \tag{4.10}$$

as above (note that the image of $\rho$ is contained in $H$ and hence the above construction of $E_H$ is possible). It is straightforward to check that for this $G$-bundle $E_G$ we have $H_0 = H$, and $E_{H_0} = E_H$ defined in (4.10).

## 5. The Levi quotient of the automorphism group

In this final section, we will assume $\Gamma$ to be a connected algebraic group. We will also assume the action of $\Gamma$ on $E_G$ to be algebraic, that is, the map $\phi$ in (2.1) is algebraic; consequently, the action of $\Gamma$ on $M$ is also algebraic. Since $\Gamma$ is connected, the action of $\Gamma$ on $\mathrm{Aut}(E_G)$ preserves the subgroup $\mathrm{Aut}^0(E_G)$.

Let $U\,\mathrm{Aut}^0(E_G)$ be the unipotent radical of the algebraic group $\mathrm{Aut}^0(E_G)$ [6, p. 125]. So the Levi quotient,

$$L\,\mathrm{Aut}^0(E_G) := \mathrm{Aut}^0(E_G)/U\,\mathrm{Aut}^0(E_G), \tag{5.1}$$

is a connected reductive algebraic group defined over $k$. Let,

$$\psi : \mathrm{Aut}^0(E_G) \to L\,\mathrm{Aut}^0(E_G), \tag{5.2}$$

be the quotient map.

From the uniqueness of a unipotent radical it follows immediately that the action of $\Gamma$ on $\mathrm{Aut}^0(E_G)$ preserves the subgroup $U\,\mathrm{Aut}^0(E_G)$. Therefore, we have an induced action of $\Gamma$ on $L\,\mathrm{Aut}^0(E_G)$.

Let $\widehat{T}_0 \subset G$ be a torus in the conjugacy class of tori of $G$ given by a maximal torus in $\mathrm{Aut}^0(E_G)$. Since any two maximal tori are conjugate, the conjugacy class of $\widehat{T}_0$ does not depend on the choice of the maximal torus of $\mathrm{Aut}^0(E_G)$. Let $\widehat{H}_0$ be the centralizer of $\widehat{T}_0$ in $G$. Setting $\Gamma = \{e\}$ in Proposition 3.3 we conclude that $E_G$ admits a reduction of structure group to $\widehat{H}_0$.

**Proposition 5.1.** *If $E_G$ admits a $\Gamma$-equivariant reduction of structure group to the Levi subgroup $\widehat{H}_0$, then the induced action of the group $\Gamma$ on $L\,\mathrm{Aut}^0(E_G)$ (defined in* (5.1)*) factors through an action of a torus quotient of $\Gamma$.*

*If $\Gamma$ is reductive and the induced action of $\Gamma$ on $L\,\mathrm{Aut}^0(E_G)$ factors through an action of a torus quotient of $\Gamma$, then $E_G$ admits a $\Gamma$-equivariant reduction of structure group to the Levi subgroup $\widehat{H}_0$.*

**Proof.** Assume that $E_G$ admits a $\Gamma$-equivariant reduction of structure group to $\widehat{H}_0$. Theorem 4.1 says that there is a maximal torus,

$$T_0 \subset \mathrm{Aut}^0(E_G),$$

which is left invariant by the action of $\Gamma$ on $\mathrm{Aut}(E_G)$. Consider $\psi(T_0)$, with $\psi$ defined in (5.2), which is a maximal torus in $L\,\mathrm{Aut}^0(E_G)$. Note that $\psi(T_0)$ is left invariant by the induced action of $\Gamma$ on $L\,\mathrm{Aut}^0(E_G)$, as $T_0$ is $\Gamma$-invariant.

Let $ZL\,\mathrm{Aut}^0(E_G) \subset L\,\mathrm{Aut}^0(E_G)$ be the center, and

$$PL\,\mathrm{Aut}^0(E_G) := L\,\mathrm{Aut}^0(E_G)/ZL\,\mathrm{Aut}^0(E_G)$$

the corresponding adjoint group. All the automorphisms of $L\,\mathrm{Aut}^0(E_G)$ connected to the identity automorphism are parametrized by $PL\,\mathrm{Aut}^0(E_G)$, with $PL\,\mathrm{Aut}^0(E_G)$ acting on $L\,\mathrm{Aut}^0(E_G)$ as conjugations.

Since $\Gamma$ is connected, we have a homomorphism of algebraic groups,

$$\rho : \Gamma \to PL\,\mathrm{Aut}^0(E_G),$$

such that the action of any $g \in \Gamma$ on $L\,\mathrm{Aut}^0(E_G)$ is conjugation by $\rho(g)$. Since the action of $\Gamma$ preserves the maximal torus $\psi(T_0) \subset L\,\mathrm{Aut}^0(E_G)$, and $q \circ \psi(T_0)$ is a maximal torus in $PL\,\mathrm{Aut}^0(E_G)$, where

$$q : L\,\mathrm{Aut}^0(E_G) \to PL\,\mathrm{Aut}^0(E_G)$$

is the projection, we conclude that $\rho(\Gamma) \subset q \circ \psi(T_0)$, where $\psi$ is defined in (5.2). (The maximal torus $q \circ \psi(T_0)$ is a finite index subgroup of its normalizer in $PL\,\mathrm{Aut}^0(E_G)$.) Therefore, the action of $\Gamma$ on $L\,\mathrm{Aut}^0(E_G)$ factors through the conjugation action of the torus $\rho(\Gamma)$.

To prove the second statement in the proposition, assume that the induced action of $\Gamma$ on $L\,\mathrm{Aut}^0(E_G)$ factors through the torus quotient $\Gamma \to T_\Gamma$. We will first show that the action of $T_\Gamma$ on $L\,\mathrm{Aut}^0(E_G)$ preserves a maximal torus.

Construct the semi-direct product $L\,\mathrm{Aut}^0(E_G) \rtimes T_\Gamma$ using the induced action of $T_\Gamma$ on $L\,\mathrm{Aut}^0(E_G)$. We take a maximal torus:

$$\widehat{T} \subset L\,\mathrm{Aut}^0(E_G) \rtimes T_\Gamma,$$

containing $T_\Gamma$ (note that $T_\Gamma$ is naturally a subgroup of $L\,\mathrm{Aut}^0(E_G) \rtimes T_\Gamma$). Finally, consider the intersection:

$$T_1 := \widehat{T} \cap L\,\mathrm{Aut}^0(E_G)$$

(note that $L\,\mathrm{Aut}^0(E_G)$ is a normal subgroup of $L\,\mathrm{Aut}^0(E_G) \rtimes T_\Gamma$). From its construction it is immediate that $T_1$ is a maximal torus of $L\,\mathrm{Aut}^0(E_G)$ and $T_1$ is left invariant by the action of $T_\Gamma$ on $L\,\mathrm{Aut}^0(E_G)$.

Consider the subgroup:

$$G' := \psi^{-1}(T_1) \subset \mathrm{Aut}^0(E_G),$$

where $\psi$ is the projection in (5.2). Since $\Gamma$ preserves $T_1 \subset L\,\mathrm{Aut}^0(E_G)$, the action of $\Gamma$ on $\mathrm{Aut}^0(E_G)$ preserves the subgroup $G'$ defined above. Note that $G'$ fits in an exact sequence:

$$e \to U\,\mathrm{Aut}^0(E_G) \to G' \to T_1 \to e,$$

where $U\,\mathrm{Aut}^0(E_G)$, as before, is the unipotent radical.

A maximal torus of $G'$ is a maximal torus of $\mathrm{Aut}^0(E_G)$, and since $\Gamma$ is connected, an algebraic action of $\Gamma$ on a torus through automorphisms is trivial. Therefore, in view of Proposition 3.3, to prove the second statement in the proposition it suffices to show that $\Gamma$ preserves some maximal torus in $G'$.

Denote by $\mathfrak{g}'$ the Lie algebra of $G'$. The action of $\Gamma$ on $G'$ induces an action of $\Gamma$ on $\mathfrak{g}'$. Let $\mathfrak{u}$ (respectively, $\mathfrak{t}_1$) be the Lie algebra of $U\,\mathrm{Aut}^0(E_G)$ (respectively, $T_1$). So the above exact sequence of groups give an exact sequence,

$$0 \to \mathfrak{u} \to \mathfrak{g}' \xrightarrow{\beta} \mathfrak{t}_1 \to 0, \tag{5.3}$$

of Lie algebras.

Let,

$$\mathcal{V} \subset \mathfrak{g}',$$

be the subspace on which $\Gamma$ acts trivially. Note that $\mathcal{V}$ is a Lie subalgebra. The action of $\Gamma$ on $T_1$ is trivial (as the automorphism group of $T_1$ is discrete and $\Gamma$ is connected). Therefore, the induced action of $\Gamma$ on $\mathfrak{t}_1$ is trivial.

Since $\Gamma$ is reductive, any exact sequence of finite dimensional left $\Gamma$-modules defined over $k$ splits, in particular, (5.3) splits. Since $\mathfrak{t}_1$ is the trivial $\Gamma$-module, we conclude that the restriction to the subalgebra $\mathcal{V} \subset \mathfrak{g}'$ of the projection $\beta$ in (5.3) is surjective.

Let $G_2 \subset G'$ be the Zariski closed subgroup generated by the subalgebra $\mathcal{V}$. Since $\Gamma$ acts trivially on $\mathcal{V}$ we conclude that $G_2$ is fixed pointwise by the action of $\Gamma$ on $G'$.

Since the projection of $\mathcal{V}$ to $\mathfrak{t}_1$ (by $\beta$ in (5.3)) is surjective, the subgroup $G_2$ projects surjectively to $T_1$. Take any maximal torus $T_2 \subset G_2$. Since the projection of $G_2$ to $T_1$ is surjective and the kernel of the projection $G' \to T_1$ is a unipotent group, we conclude that $T_2$ is a maximal torus of $G'$.

In other words, $T_2$ is a $\Gamma$-invariant maximal torus of $G'$. Since a maximal torus in $G'$ is a maximal torus in $\mathrm{Aut}^0(E_G)$, Proposition 3.3 completes the proof of the proposition. $\square$

The following is an immediate corollary of Proposition 5.1.

**Corollary 5.2.** *If $\Gamma$ does not have a nontrivial torus quotient (for example, if it is unipotent or semisimple), and the action of $\Gamma$ on $L\,\mathrm{Aut}^0(E_G)$ is nontrivial, then $E_G$ does not admit any $\Gamma$-equivariant reduction of structure group to $\widehat{H}_0$, provided $\widehat{H}_0 \neq G$.*

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